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On the structure of nonsemisimple Hopf algebras

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Abstract. The left module structure of finite-dimensional quantum algebras is analysed using the theory of primitive idempotents. In particular, a complete structural result in terms of principal (projective) indecomposable modules (p.i.m.) is given in the case of $U_q sl(2)$ (q a root of 1) by finding a complete set of primitive idempotents. The structure of p.i.m. is analysed in detail. The Jacobson radical of the algebra is investigated and its significance in the study of nonsemisimple symmetries in physical systems is discussed.

1. Introduction

This, and the following paper, aim to investigate the structure and representation of certain finite-dimensional nonsemisimple Hopf algebras. The study of such algebras is complicated by the fact that even the finite-dimensional modules are not necessarily completely reducible. Thus, the tensor product of two irreducible modules may have some indecomposable summands. The motivation for this investigation comes from their application to physics. First, indecomposable modules appear quite frequently in physics, an early example being the indefinite metric space of Gupta and Bleuler. Similarly, in the case of quantized non-Abelian gauge theory, an indefinite metric ambient space ensures covariance and renormalization. The global symmetry algebra we are concerned with is the BRS algebra [1]—a nonsemisimple algebra. More recently, we have the fascinating example of quantum algebras at the root of 1. These algebras can be considered as 'symmetry' algebras of certain models (anisotropic Heisenberg chain [2], chiral Potts model etc). Moreover they have a deep connection with certain categories of modules of Kac–Moody algebras and Virasoro algebra and hence with conformal field theories. We are thus motivated to study the rich structure of finite-dimensional algebras derived from $U_q g$ by specializing q to a root of 1 [3]. This paper deals with the structural theory and the following paper studies representations using the structural results. After giving some general results, valid for any finite-dimensional Hopf algebra I, we concentrate on $U_q sl(2)$. The results are complete in this case. Some of these results were announced earlier [4]. I sketch the proofs since they involve some interesting applications of basic hypergeometric series. The (Jacobson) radical is explicitly given in terms of generators and is used to show that the results are indeed complete. The radical is used to single out the 'physical' state space. The similarities with some earlier results on indecomposable representations of Lie groups on a space with an indefinite metric [5] are also discussed.

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2. Structural theory

Let us recapitulate some well known results from the theory of finite-dimensional associative algebras [6]. Most of these are true for general Artinian rings. Let U be such an algebra. As a left module $U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ where each U_i is indecomposable and by the Krull–Schmidt theorem this decomposition is unique up to an isomorphism. In the case of semisimple U each p.i.m. U_i is irreducible. When U is nonsemisimple, however, this is not generally true. Nevertheless, their importance lies in the fact that any projective U-module is isomorphic to a direct sum of p.i.m. An element $e \in U$ is an idempotent if $e^2 = e$ and a primitive if e cannot be written as a sum of orthogonal idempotents, e_1 and e_2. A basic result is that a left U-ideal is a p.i.m. if and only if $I = Ue$ for some primitive idempotent e. Thus

$$U = Ue_1 \oplus Ue_2 \oplus \cdots \oplus Ue_r \tag{1}$$

for some primitive idempotents $e_1, e_2, \ldots, e_r$. Some of the components may be isomorphic. However, each Ue_i is indecomposable and being a direct summand of U is projective. Let R denote the radical of U. Then any irreducible U-module is isomorphic to Ue/Re for some primitive idempotent e. Finally, if U happens to be Frobenius (i.e. possessing an associative bilinear form) then the following holds.

Theorem 1. *If the ground field $\mathcal{F}$ is a splitting field for a Frobenius algebra U then the number of times Ue (e primitive idempotent) appears as a component of U is equal to the $\mathcal{F}$-dimension of Ue/Re.*

All the facts cited above are given in the Curtis–Reiner classic [6].

Next, let g be a simple Lie algebra and $U_q g$ the corresponding quantum algebra. Specializing q to a root of 1 produces a 'remarkable' finite-dimensional Hopf algebra $\tilde{U}_q g$. The details can be found in [3]. Moreover, this finite-dimensional algebra can be defined via generators and relations. Since any finite-dimensional Hopf algebra possesses an integral (unique up to a multiple) $\tilde{U}_q g$ is Frobenius [7]. In fact it is symmetric but we do not need this. Thus, theorem 1 is applicable. Here we write explicitly only for $g = sl(2)$. Throughout, q will be assumed to be a primitive pth root of 1, p odd. Let A be an associative algebra with generators E, F, and K and defining relations

$$KEK^{-1} = q^2 E \qquad KFK^{-1} = q^{-2} F \tag{2}$$

$$[EF] = \frac{K - K^{-1}}{q - q^{-1}} \tag{3}$$

$$E^p = F^p = 0 \qquad \text{and} \qquad K^p = 1. \tag{4}$$

The algebra A is defined over $Q(q)$ but we will treat it as an algebra over $\boldsymbol{C}$. The relations (4) imply that $\dim(A) = p^3$. The rest of the Hopf algebra structure is given by

$$\triangle(K) = K \otimes K \tag{5}$$

$$\triangle(E) = E \otimes 1 + K \otimes E \qquad \triangle(F) = F \otimes K^{-1} + 1 \otimes F \tag{6}$$

$$\epsilon(K) = 1 \qquad \epsilon(E) = \epsilon(F) = 0 \tag{7}$$

$$S(K) = K^{-1} \qquad S(E) = -K^{-1}E \qquad S(F) = -FK. \tag{8}$$

It is also known that for any Hopf algebra the tensor product of a projective module with any module is projective. This is another reason why p.i.m. are of importance in the study of structure and representations of nonsemisimple algebras.

We now fix some notation:

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}} \qquad [a]! = [a] \cdot [a-1] \ldots [1]$$

$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{[n]!}{[j]! \cdot [n-j]!} \qquad \text{and} \qquad [\alpha]_r = \prod_{m=0}^{r-1} \frac{(\alpha q^m - \alpha^{-1} q^{-m})}{q - q^{-1}}.$$

The last one is not conventional but it makes the expressions compact.

Lemma 1. *For $m = 0, \ldots, p-1$ let*

$$x_m = aE^{p-1} \sum_{r=0}^{m} F^{p-1-r} \phi(s_r) \tag{9}$$

where $\phi(\lambda) = 1 + \lambda + \lambda^2 + \cdots + \lambda^{p-1}$ and $s_r = Kq^{2r+1}$. Then x_m are primitive idempotents.

Proof. We need the following identity due to Kac [8]:

$$[E^m, F^n] = \sum_{j=1}^{\min(m,n)} [j]! \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} F^{n-j} \prod_{r=j-m-n+1}^{2j-m-n} [K; r] E^{m-j} \tag{10}$$

where $[K; r] = (Kq^r - K^{-1}q^{-r})/(q - q^{-1})$.

First, to show that $x_m^2 = x_m$ note that a typical summand in x_m^2,

$$E^{p-1} F^{p-1-r} \phi(s_r) E^{p-1} F^{p-1-t} \phi(s_t) = 0 \qquad \text{for} \quad r > 0.$$

Thus we have to consider only terms with $r = 0$. Then

$$\begin{aligned} E^{p-1} F^{p-1} \phi(s_0) E^{p-1} F^{p-1-t} \phi(s_t) &= [p-1]! E^{p-1} F^{p-1-t} \prod_{m=0}^{p-2} [K^{-1}; m - 2t] \phi^2(s_t) \\ &= \frac{p^2}{(q - q^{-1})^p} [p-1]! E^{p-1} F^{p-1-t} \phi(s_t). \end{aligned} \tag{11}$$

The last equality holds because $K\phi(s_t) = q^{-2t+1}\phi(s_t)$ and the first one is an easy consequence of (10) and nilpotency of E. Since $[p-1]! = p/(q - q^{-1})^p$ we get the normalization constant

$$a = (p[p-1]!^2)^{-1}. \tag{12}$$

Next, to show that x_m is primitive let us suppose that we have two idempotents a_m and b_m such that

$$x_m = a_m + b_m \qquad a_m b_m = b_m a_m = 0.$$

Let

$$a_m = E^{p-1} \sum_{r=1}^{p} F^{p-r} h_r(K) + E^{p-2} \sum F^{p-r} h'_r + \cdots. \tag{13}$$

In a product $E^{r_1} F^{s_1} E^{r_2} F^{s_2}$ written in the chosen basis $E^i F^j K^r$ the minimum power of $E \geqslant r_1$ and that of $F \geqslant s_2$. Therefore, from the condition $x_m a_m = a_m$ one concludes that only the first term in the above equation survives. An argument like the one used in proving the idempotency of x_m shows that the polynomials $h_r(K)$ are uniquely determined. Hence $a_m = x_m$. □

Consider now the p.i.m. generated by x_m as follows.

Theorem 2. *The p.i.m. Ax_m are cyclic modules generated by x_m with highest weight q^{-1} and x_m is an E-singular vector. Hence $F^k x_m$ form the respective bases. Moreover, they are mutually isomorphic and irreducible. The common dimension is p.*

All the facts stated in the theorem are straightforward consequences of the structure of x_m. Let P_1 denote the isomorphism class of Ax_m. P_1 is the only such class of a projective module which is irreducible. This corresponds to the Steinberg module of classical modular Lie algebras [9]. One can show, as in the classical case, that each irreducible module V_r is cyclic, generated by a unique E-singular vector with weight q^r, $r = 0, \ldots, p-1$. Since A is Frobenius we can use theorem 1. The members of class P_1 being irreducible are annihilated by the radical R. Hence there are exactly p such modules and Ax_m, $m = 0, \ldots, p-1$ being disjoint, exhaust these. We can thus write

$$A = \sum_{A_i \in P_1} A_i \oplus N \tag{14}$$

where N is the sum of the rest of the p.i.ms. To study N we need to know more about the structure of the radical.

Theorem 3. *The (Jacobson) radical R of A is a two-sided ideal generated by $E^{p-1}[K;-1]$ and $F^{p-1}[K;1]$. R^2 is a two-sided ideal generated by $E^{p-1}F^{p-1}[K;1]$ and $R^3 = 0$.*

Proof. First note that the generators $E^{p-1}[K;-1]$ and $F^{p-1}[K;1]$ annihilate all irreducible left A-modules which follow easily from their structure. The fact that they generate R can be shown by induction on r for a homogeneous element $\sum_{s=0}^{r} E^{p-1-s}F^{r-s}f_s(K)$. The assertions regarding R^2 and R^3 are simple consequences of the form of the generators. □

The next lemma gives some more primitive idempotents.

Lemma 2. *For $2 \leqslant n \leqslant p$ let*

$$y_n = a_n \sum_{k=0}^{n-2} (-1)^k \frac{E^{p-n+k}F^{p-n+k}\phi(Kq^n)}{[k]![p-n+k+1]!} + \frac{E^{p-1}F^{p-1}\phi(Kq^n)}{[n-1]!}. \tag{15}$$

Then for a suitable choice of a_n, y_n are primitive idempotents.

Proof. From a simple application of (10) we get the following useful identity

$$E^rF^rE^sF^s = \sum_{j=0}^{\min(r,s)} \begin{bmatrix} r \\ j \end{bmatrix} \begin{bmatrix} s \\ j \end{bmatrix} [j]! E^{r+s-j}F^{r+s-j} \prod_{m=1}^{j} [K^{-1}; m+r+s-j]. \tag{16}$$

First note that y_n^2 contains only terms in which the exponents of E and F are equal. Now consider the terms in y_n^2 that can yield the term $E^{p-n+k}F^{p-n+k}\phi(Kq^{-n})$. Letting

$$a_k = \frac{(-1)^k}{[k]![p-n+k+1]!}, \quad k < n-1 \qquad \text{and} \qquad a_{n-1} = \frac{1}{[n-1]!}$$

the terms mentioned are

$$\Bigg(\sum_{r=0}^{k} (a_{k-r}E^{p-n+k-r}F^{p-n+k-r})^2 + 2\sum_{s=1}^{k-r} a_{k-r}a_{k-r-s}E^{p-n+k-r}F^{p-n+k-r}E^{p-n+k-r+s}F^{p-n+k-r+s} \Bigg)\phi^2. \tag{17}$$

We show that the coefficient of $E^{p-n+k}F^{p-n+k}\phi(Kq^{-n})$ obtained from the above expression is proportional to a_k and the constant of proportionality independent of k. We observe first that $\phi^2(Kq^{-n}) = p\phi(Kq^{-n})$. Next, expanding (17) and using (16), we get the coefficient of $E^{p-n+k}F^{p-n+k}\phi$:

$$c_k = \sum_{r=0}^{k} a_{k-r}^2 [p-n+k-2r]! \begin{bmatrix} p-n+k-r \\ p-n+k-2r \end{bmatrix} \prod_{m=1}^{p-n+k-2r} [q^n;\, p-n+k+m]$$
$$+pa_{k-r} \sum_{s=-r}^{k-r} a_{k-r-s}[p-n+k-2r-s]! \begin{bmatrix} p-n+k-r \\ p-n+k-2r-s \end{bmatrix}$$
$$\times \begin{bmatrix} p-n+k-r-s \\ p-n+k-2r-s \end{bmatrix} \prod_{m=1}^{p-n+k-2r-s} [q^n;\, p-n+k+m]. \tag{18}$$

Note that for any positive integer m

$$[p-1-m]! = (-1)^m p/[m]!. \tag{19}$$

Using this identity in the expression for a_{k-r+s} and changing the summation index we get

$$c_{kr} = \frac{(-1)^{r+1}p[p-n+k-r]!}{[k]![r]!}$$
$$\times \sum_{u=0}^{k} (-1)^s \frac{[n-k+u-2]![n-k+r+u-1]!}{[u]![k-u]![n-k+u-1]![n-2k+r+u-1]!}.$$

This can be rewritten as

$$c_{kr} = -\frac{p[p-n+k-r]![n-k-2]![n-k+r-1]!}{[k]![r]![n-k-1]![n-2k+r-1]!}$$
$$\times \sum_{u=0}^{k} \frac{[q^{n-k-1}]_u [q^{n-k+r}]_u [q^{-k}]_u}{[u]![q^{n-k}]_u [q^{n-2k+r}]_u}. \tag{20}$$

We can now apply the q-Saalschütz identity which is a summation formula for a balanced terminating ${}_3\phi_2$ basic hypergeometric series [10]. The symmetric version of this formula is

$$\sum_{n=0}^{N} \frac{[a]_n [b]_n [q^{-N}]_n}{[n]![c]_n [abc^{-1}q^{1-N}]_n} = \frac{[c/a]_N [c/b]_N}{[c]_N [c/ab]_N}. \tag{21}$$

Setting $a = q^{n-k-1}$, $b = q^{n-k+r}$ and $c = q^{n-k}$ in (20) we obtain

$$c_{kr} = -\frac{p[p-n+k-r]![n-k+r-1]![n-k-2]![k]!}{([k]!)^2[r]![n-k-1]![n-k+2r-1]!} \frac{[q^{-r}]_k}{[q^{n-k}]_k [q^{-n+k-r+1}]_k}. \tag{22}$$

This sum vanishes for all $0 \leqslant r < k$. Thus the only term contributing to the sum (20) is $r = k$. Hence

$$c_k = (-1)^k \frac{([p-n]!)^2}{[k]![p-n+k+1]![p-n+1]!}. \tag{23}$$

Thus $a_n = (-1)^{n-1}([p-n]!)^2/[p-n+1]!$ in (15) and the lemma is proved. □

We next give the other primitive idempotents which generate left modules that are isomorphic to those of y_n. The left module structure is unaffected whether we consider an idempotent e, or a scalar multiple we loosely call y_n, and any such multiples idempotents. Some more primitive idempotents are given by the following.

Lemma 3. *Let $y_{n0} = y_n$ and for $1 \leqslant k \leqslant p-n$*

$$y_{nk} = y_n + \sum_{r=1}^{n} a_{p-r} E^{p-r} F^{p-r-k} \phi(q^{n+2k}K) \tag{24}$$

with

$$a_{p-n+k} = (-1)^k \frac{[p-n+1]!}{[k]![p-n+k+1]![p-n]!^2} \tag{25}$$

equal to the correspnding coefficient in y_n. *Then* y_{nk} *are primitive idempotents and the map* $y_n \to y_{nk}$ *extends naturally to an isomorphism of left* A*-modules* Ay_n *and* Ay_{nk}.

Proof. Letting $z_{nk} = y_{nk} - y_n$ we have $z_{nk}^2 = z_{nk} y_n = 0$. Hence to prove that $y_{nk}^2 = y_{nk}$ we have to show $y_n z_{nk} = z_{nk}$. This is proven as before with another appeal to Saalschütz! A similar proof for primitivity holds. The last statement is shown by simply proving that for $x \in A$, $xy_n = 0 \Leftrightarrow xy_{nk} = 0$. □

Note that if we can make sense of F^{-1} (even though $F^p = 0$) then formally $y_{nk} = y_n + y_n F^{-k}$. Now let us investigate the structure of the isomorphism class of modules $\{Ay_n\}$. Let

$$\alpha_n = E^{n-1} y_n \qquad \text{and} \qquad \beta_n = F^{p-n+1} y_n. \tag{26}$$

Lemma 4. *The module* Ay_n *is generated as a vector space by* $F^k y_n$ *and* $F^k \alpha_n$. *There are two E-singular vector* α_n *and* $F^{n-1}\alpha_n$ *with weights* q^{n-2} *and* q^{-n} *respectively. Moreover*

$$Ey_n = ([n-2]!)^{-2} F^{n-2}\alpha_n \qquad E\beta_n = ([n-2])^{-2} F^{p-1}\alpha_n \tag{27}$$

$$FEy_n = a_{p-2} E^{p-1} F^{p-1} \qquad \textit{and} \qquad \beta_n = cE^{n-2} F^{p-1} \phi(q^n K) \tag{28}$$

where c *is some nonzero constant. For all* $0 \leqslant k \leqslant p-1$, $F^k \alpha_n$ *and* $F^k y_n$ *are not zero and the respective subspaces spanned by them are disjoint. In particular* $\dim\{Ay_n\} = 2p$.

Proof. It is easily verified that $\alpha_n = a_{p-n} E^{p-1} F^{p-n} \phi(q^n K)$ and $E\alpha_n = 0$. Moreover, since $[EF^{n-1}] \cdot \alpha_n = 0$, $EF^{n-1}\alpha_n = 0$. Again $[EF^{p-n+1}] \cdot y_n = 0$ implies the second formula. The third equation follows by direct calculation since all terms except $E^{p-1}F^{p-1}\phi$ cancel and the last one is a consequence of $EF^{p-n+1} y_n = F^{p-n+1} E y_n$ and the preceding formulae. It is clear that $F^k \alpha_n$ and $F^k y_n$ are nonzero for $0 \leqslant k \leqslant p-1$. By using induction and the fact that only for eigenvalues $q^{-n}, q^{-n-2}, \ldots, q^{-n-2(p-n)}$ of K do the corresponding eigenspaces in $F^{k+n-1}\alpha_n$ and $F^k y_n$ overlap, we prove the disjoint property stated in the lemma. □

We now have a clear picture of the structure of p.i.m. Rescaling we may assume that $Ey_n = F^{n-2}\alpha_n$. Let Q_n denote a representative of the isomorphism class $\{Ay_n\}$. Then Q_n has a basis $\{F^k y_n, F^k \alpha_n\}$. The A-submodule Ry_n generated by the radical is spanned by $F^k \alpha_n$, $0 \leqslant k \leqslant p-1$ and $F^k y_n$, $p-n+1 \leqslant k \leqslant p-1$. The dimension of the irreducible module Ay_n/Ry_n is thus $p-n+1$. But we have already discovered a set of primitive idempotents $\{y_{nk}\}$ for $0 \leqslant k \leqslant p-n$ and clearly the corresponding modules are independent, i.e. $Ay_{nk} \bigcap (Ay_{n0} \oplus \cdots \oplus Ay_{nk-1}) = 0$. Let x_m be as defined in (9). The complete result on the left module structure of A is now summed up.

Theorem 4. *Let* x_m *and* y_{nk} *be defined as above (9) and (24). Then*

$$A = \sum_{m=0}^{p-1} Ax_m \oplus \sum_{n=2}^{p} \sum_{k=0}^{p-n} Ay_{nk} \tag{29}$$

where all the sums are direct. The primitive idempotents x_m *generate isomorphic* A*-modules of dimension* p. *For fixed n the modules* Ay_{nk} *are isomorphic and the common dimension is* $2p$. *Any p.i.m. of* A *is either of the type* Ax_m *(*P*-type) or* Ay_n *(*Q*-type).*

The modules of type P or Q_n can be obtained by the appropriate tensoring [11, 12]. In fact one starts with P, which can directly be shown to be projective, and then takes the product $P \otimes V_m$ so that the corresponding indecomposable components will yield the modules Q_m. However, we cannot be sure that these exhaust all the p.i.m. In the case of higher algebras p.i.m. can be generated by tensoring with the Steinberg module. Chari and Premet [13] used different methods to classify Weyl modules and their projective covers. The Weyl modules can be obtained from the p.i.m. by applying the radical.

In conclusion let us note the following facts. Since there are embeddings of $U_q sl(2)$ in $U_q sl(n)$ the primitive idempotents carry over to the latter. This could be a starting point for the study of higher-dimensional algebras by combining other techniques such as the Gelfand–Tsetlin construction or canonical bases [14]. Moreover, the radical gets embedded in the radical of the latter. It is suggested that the radical or a larger primitive ideal will have an important bearing on nonsemisimple symmetries in physical systems. To illustrate this let us recall from theorem 3 that the radical R of $U_q sl(2)$ has the property that $R^3 = 0$ and $R \cdot M = 0$ for any irreducible module M. We could characterize the irreducible modules by this condition. This explains the two characterizations of type I singular vectors in [2]. When we consider multiparticle states of a system with $U_q sl(2)$ symmetry tensor products of irreducible modules become necessary. We may, therefore, have some indecomposable components. To eliminate states corresponding to the latter we may use the radical characterization. This could provide an alternative to truncated tensor products as defined in the literature [11].

Finally, we may note that the indecomposable modules may be of interest to physics for two reasons: (i) any contravariant bilinear form on such a module must necessarily be indefinite and (ii) there may be metastable or unstable states of physical systems with nonsemisimple symmetries which are described by indecomposable modules. In fact there are many more models with indefinite metric space than mentioned above (see [5] and references therein). For the Q-type p.i.m. that we have considered it is not difficult to define an essentially unique indefinite contravariant form. Now consider the filteration

$$Q_n \supset R \cdot Q_n \supset R^2 \cdot Q_n \supset R^3 \cdot Q_n = 0.$$

With respect to the indefinite form the irreducible module $R^2 \cdot Q_n$ is orthogonal to $R \cdot Q_n$ (this implies $R^2 \cdot Q_n$ consists of zero norm states). Araki's results on indecompsable representations [5] are valid in this case. In fact, the three nonzero modules in the above filteration define a Gupta–Bleuler triplet. The paper of Araki gives a cohomolgical treatment of such chains for indecompsable representations of Lie groups. Such an analysis may be carried out here. Furthermore, in some problems we may have to consider primitive ideals larger than the radical to restrict the space of states to a smaller subclass than the irreducible modules.

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